Fractal characteristics of lakes

Background

Evaluating the abundance, size, and shape of lakes are among the most fundamental activities of scientists studying Earth's inland waters. However, relatively little is known about these characteristics at the global-scale because most limnological studies only consider one lake or a few lakes in close proximity, and because there is critical lack of testable theory to guide analyses and data collection at broad geographic scales (Seekell et al. 2013).

A large body of literature indicates that Earth's topography is approximately self-affine (e.g. Mandelbrot 1975, Matsushita et al. 1991, Russ 1994). This means that rescaling topographic profiles horizontally results in predictable scaling of the vertical profile. The Hurst exponent $H$ characterizes the relationship between horizontal scaling and vertical scaling. In principal, the range of Hurst exponent is $0 \leq H \leq 1$, with $H=1$ being a special case known as self-similarity and $H=0$ being white noise (Goodchild 1988, Ouchi & Matsushita 1992). Self-similar surfaces appear the same regardless of scaling. Earth's topography is not self-similar, and this is self-evident when viewing mountain profiles, which flatten if the viewing point is moved away from the mountain range (Ouchi and Matsushita 1992). It is also self-evident that topography is not white noise because patterning is clearly visible on Earth's surface when examining a map or looking out an airplane window. Empirical estimates of the Hurst exponent for Earth's topography vary, in part based on study extent and methods, but are typically between 0.3 and 0.7 (Goodchild 1988, Cael et al. 2017).

The predictable scaling of self-affine surfaces has been the basis for developing hypotheses about diverse geographic features includ-
ing river networks and the ocean floor, and is a promising approach to improving understanding of global lake morphometry. Specifically, self-affine surfaces are fractal, and fractal dimensions for geographic phenomena are useful for testing these hypotheses because they are readily measured (Goodchild 1988). Here, I summarize connections between the properties of self-affine surfaces and lake morphometry. I highlight areas where these connections have improved scientific understanding of Earth's lakes, and identify areas where progress has been limited.

**Abundance, surface area, and shore length**

How many lakes are there and how big are they? Do large lakes or small lakes comprise most of Earth's lake surface area? These are

![Figure 1](image_url)

*Figure 1. The figure shows that probability of small lakes being included in United States Geological Survey 1:100,000 maps decreases for small lakes. The quality of these 1:100,000 scale maps exceeds the quality of maps available in most regions globally. Peter Lake and Paul Lake are experimental lakes located in Michigan, USA. These lakes have contributed disproportionately to limnologists understanding of all lakes including seminal tests of biological control of primary production and greenhouse gas emissions, diversity of basal energy sources for aquatic food webs, and early warning indicators for ecosystem regime shifts. Despite this, lakes of this size have less than a coin-flip chance of being counted in national maps, and no chance of inclusion in global maps like the widely used map compiled by Lebner and Doll (2004).*
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...some of the most basic questions about lakes, and also some with the most poorly constrained answers (Seekell et al. 2013). This uncertainty is a direct result of small lakes being omitted from maps in many regions. Even the best compilations of global lake data are thought to greatly underestimate the abundance and surface area of small lakes (Figure 1). This is problematic because small systems contribute disproportionately to scientific understanding of lakes, and because small lakes may process elements or store elements at globally significant rates (Downing 2010).

The characteristics of self-affine surfaces can contribute to understanding the abundance and surface area of lakes at the global scale. Specifically, the fractal dimension of a landscape ($D_{\text{surface}}$) is $D_{\text{surface}}=2-H$, indicating that landscapes in the normal range of Hurst exponents are rugged. Intersecting a horizontal plane at the mean elevation creates a set of points where the surface intersects the plane. These points can be connected to form contour lines. Because lake shorelines are just the contour lines for the lake surface elevation, the area enclosed by points is analogous to lake areas (in the sense of Matsushita et al. 1991; Russ 1994). The collection of areas enclosed by the shorelines are power law distributed and associated with a fractal dimension that represents the convoluted nature of the collection of shorelines, which itself directly reflects the ruggedness of the landscape (Matsushita et al. 1991; Isogami & Matsushita 1992; Russ 1994; Sasaki et al. 2006). The fractal dimension of the size distribution of lakes is measured with the regression:

$$N = c A^{-(D/2)}$$

where $N$ is the number of lakes greater than or equal to the area $A$, $c$ is a constant, and the functional form (i.e., power law) of the regression is based on the first return rate of a fractional Brownian to the mean elevation (Goodchild 1988; Matsushita et al. 1991). $D$ is the fractal dimension of the shorelines surrounding the lake area and is one less than the dimension of the landscape. $D$ is constrained between $D=1$ (a population of perfectly smooth shorelines) and $D=2$ (a population of shorelines so irregular they are space filling). Hence, there is a theoretical basis for a power law size-distribution and theoretical constraints $0.5 \leq (D/2) \leq 1$ on the plausible range of exponents (Goodchild 1988; Hamilton et al. 1992).
A primary conclusion that can be drawn from these results is that, because $(D/2)<1$, the vast majority of Earth’s lakes are small, but large lakes contribute disproportionately the total surface area (Seekell et al. 2013). These results can also be used to draw conclusions about the total abundance and surface area of lakes by fitting the abundance-area relationship to surface areas of large, well-mapped lakes, and then extrapolating the numbers and area of small unmapped lakes (e.g. Downing et al. 2006). However, this extrapolation is a perilous process. Different extrapolations have resulted in very different estimates of lake abundance and area – from about 8.5 million lakes covering 2.8 million square kilometers (Meybeck 1995), to about 314 million lakes covering 4.2 million square kilometers (Downing et al. 2006). This is in part because of inconsistency over minimum size to extrapolate to, and in part because of strong sensitivity of statistical methods for empirically estimating of the power exponent to slight variations in data (Seekell and Pace 2011). Extrapolation based on theoretical power-exponents from independent measures of the fractal dimensions of landscapes is possible, but is probably ill advised because the abundance-area relationship may deviate from power-law form for small lakes if scale-dependent geomorphic processes cause break-downs in the self-affine nature of the landscape at small scales (Cael and Seekell 2016). The consequence of this would be unknowingly overestimating the abundance and area of small lakes, although it would not change the overall result of dominance of small lakes by number and large lakes by area (Seekell & Pace 2011; Cael & Seekell 2016). This problem effects extrapolation based both on theoretical and empirical fractal dimensions (cf. Cael & Seekell 2016).

A secondary consequence of these results relates to lake shorelines. Specifically, because the dimension of lake shorelines exceeds unity, they are not easily measured because changing the unit of measurement alone will result in different measurements of shore length for the same lake (cf. Mandelbrot 1967). This is problematic because widely used measures of the planar shape of lakes are based on measures of shore length. Specifically, the shoreline development index, which is calculated as the shore length divided by the circumference of a circle with the same area as the lake, is biased and will increase for lakes with larger surface areas even if the shape does not change (Kent & Wissmar 2010).

**Mean depth and volume**

The mean depth and volume of lakes are used by limnologists because they are tied to the spatial distribution of different ways of nutrient and organic carbon uptake by benthic and epilithic habitat. Mean depth and volume are often measured from bathymetric surveys conducted over large spatial scales (Hollister et al. 2015). The more recent approaches have been generated for which volume and depth are often derived from spatially interpolated topographic charts that are area specific (Cael and Seekell 2016)

Fractal geometric depth and volume scales (z$_{mean}$) in the landscape (z$_{mean}$) because these scales are derived from lake depth and volume measurements (Cael and Seekell 2016)

These results have important implications for the study of plausible expected changes in lake area and volume under warming across regions and at the local scale.
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change (Kent & Wong 1982). These metrics should be used with caution in empirical analysis until a normalized version is developed and tested.

Mean depth and volume

The mean depth and volume of lakes are broadly interesting to limnologists because they relate to water residence times, rates and pathways of nutrient cycling, and the light environment for algal and fish habitat. Mean depth and volume measurements require detailed bathymetric surveys, which are expensive and time consuming to conduct (Hollister & Milstead 2010). A growing number of regional reports use lake area and lakeside topography (e.g. maximum land slope in a 50 m wide buffer around the lake shoreline) to predict individual lake depth and volume (e.g. Sobek et al. 2011; Heathcote et al. 2015). This information is readily derived from maps, and these approaches have the potential to greatly expand the number of lakes for which volume and lake depth is known. However, the prediction accuracy for these approaches is highly variable. Additionally, the topographic characteristics identified as the best correlates for depth and volume vary regionally. This suggests that these methods are region specific and not generalizable to the global scale (Oliver et al. 2016).

Fractal geometry provides some insights into patterns of mean depth and volume. Specifically, reports indicate that the mean depth \( z_{\text{mean}} \) scales with surface area based on the Hurst exponent of the landscape \( z_{\text{mean}} \sim A^{H/2} \) (Cael et al. 2017). This is a useful observation because mean depth relates volume \( V \) to surface area \( V \sim z_{\text{mean}} A \). Consequently, volume also scales with surface area \( V \sim A^{1+H/2} \) (Cael et al. 2017). An evaluation of volume-area scaling relationships for lakes from 22 biophysical regions found that the relationship was consistent across regions and with independently measured values of Hurst exponents (Cael et al. 2017). Additionally, the distribution of residuals around the relationship was consistent across the size spectra.

These results have a few important implications. First, the range of plausible exponents for the volume-area relationship indicates that almost all of Earth’s lake volume is contained in just a few large lakes.
Specifically, just the 20 largest lakes by surface area contain about 80 per cent of Earth's lake water (Cael et al. 2017). Small lakes contribute negligibly to the total volume. Second, the consistent scaling coefficient and residual uncertainty across regions allow estimation of the global lake volume with confidence intervals. A recent study estimated approximately 199,000 km$^3$ of total lake volume with the 95 per cent confidence interval 196,000-202,000 km$^3$ (Cael et al. 2017). A final implication of the fractal results is that that global lake volume is better constrained than lake abundance or area, despite the number of bathymetric measurements being on the order of tens of thousands compared to tens of millions to hundreds of millions for abundance and area. This is a direct result of the fractal scaling coefficients, which indicate that almost all lake volume is in just a few lakes, all of which have well known volumes. The variation in small lakes that greatly impacts global estimates abundance and surface area (see above) does not strongly impact volume. For example, volume estimates based on global lake databases with $2.43 \times 10^6$ km$^2$ and $5 \times 10^6$ km$^2$ surface area produce volume estimates that only differ by 15,000 km$^3$ (Cael et al. 2017).

The primary limitation of the fractal approach is that it describes characterizes a large population of lakes by predicting the slope of a scaling relationship, but it is not intended for predicting the volume and mean depth of any individual lake and it cannot make this predictions accurately. This limitation does not strongly impact evaluations of lake abundance or area because the contemporary questions in about these characteristics revolve around the global population of lakes. In contrast, most questions related to volume and mean depth are formulated on the scale of individual lakes, and accurate volume and depth measurements are needed to form the basis for other analyses. Hence, fractal geometry provides important global context for limnological studies but cannot substitute for bathymetric surveys for many limnological questions.

**Maximum depth and basin shape**

The maximum depth of lakes is interesting to limnologists because it relates strongly to patterns of thermal stratification, which in turn...
relate to hypolimnetic oxygen depletion, the volumetric extent of oxygenated fish habitat, and likelihood of winterkill (Oliver et al. 2016). Similar to mean depth, maximum depth is typically measured through bathymetric surveys, which means that maximum depth is known only for a limited number of lakes globally. Also similar to mean depth, there is a growing body of literature that seeks to predict maximum depth with statistical models calibrated to topographic data (e.g. Sobek et al. 2017; Heathcote et al. 2015; Oliver et al. 2016). Typically, there is limited practical use for these models because of large prediction errors.

Many of Earth’s largest lakes, like Lake Baikal and the Caspian Sea, are also among its deepest so it is natural to expect that maximum depth should scale with lake area. However, there is no clear connection between the maximum depth of individual lakes and the characteristics of fractal surfaces. Specifically, maximum depths reflect a single random displacement from a topographic profile and the magnitude of a single random displacement may not be predictable in any practical way. This is in contrast to statistical characteristics like mean depth or variance of depth that integrate over many random displacements and can be connected directly to Hurst coefficient and fractal dimension of the landscape (Ouchi & Matsushita 1992; Cael et al. 2017). This random nature explains why studies seeking to predict maximum depth based on topographic characteristics consistently fail to generate predictive power.

The cross-sectional basin shape is of interest to limnologists and is generally described with the mean to maximum depth ratio. With some assumptions, the mean to maximum depth ratio describes the relative size of near-shore and open-water habitats (Carpenter 1983). Inability to predict mean or maximum depth for individual lakes means that fractal approaches are not useful for estimating this metric of basin shape. If there were a clear scaling relationship for maximum depth, this could be combined with the scaling relationship for mean depth described above to derive hypothesis about potential patterns in basin shape, for example relative to surface area. However, the current disconnect between fractal geometry and maximum depth in lakes precludes generation of hypotheses about broader population characteristics.
Questions for future research

The characteristics of self-affine surface are most useful to limnologists when the geographic scale of limnological questions is large and includes many lakes. For an example, a recent study estimates the proportion of lakes on the Swedish landscape that fall within a particular mean depth interval based on the fractal volume-area scaling relationship (Seekell et al. 2018a). The study cannot (and does not attempt to) identify the specific lakes within the depth range of interest but can reliably identify the number of lakes in this range, and a similar analysis could be conducted for surface area or volume. Hence, there is strong potential for fractal scaling relationships to generate regional context for smaller-scale studies and this would be particularly useful for hard to measure characteristics like volume and mean depth.

Because many key lake characteristics scale with area, volume, or mean depth, the asymmetry between size and abundance is an ultimate constraint on global patterns in lake ecology (Cael & Seekell 2016). However, there is almost no research on the factors creating regional variation in lake size distributions, specifically for surface area for which regional variation has been reported (Downing et al. 2006). One reason for this could be differences in the magnitude of vertical relief between regions (Seekell et al. 2013). Specifically, most abundance-size relationships for geographic features are based on the assumption that all features are at the same elevation – like oceanic islands.

This assumption does not hold for lakes and it is possible that vertical relief could preclude the formation of large lakes and alter the lake size spectra (Seekell et al. 2013). In the future this could be tested through computer simulation (e.g. Goodchild 1988), or potentially tested empirically by comparing the distributions of lakes and islands on individual lakes. The perimeters of both lakes and islands should share fractal dimensions with the landscape, but islands are at a common elevation while lakes are not. Comparisons of lake distributions between regions are probably less useful for evaluating these fundamental hypotheses because lake distributions in many regions may reflect human activities as much as they do the natural topography.
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(e.g. Steele & Heffernan 2017). The effect of the overall size of study extents could also be evaluated in the context of lakes using computer simulation. Analogous studies have found that the finite area of surfaces can cause deviations from theoretical scaling laws, which are typically based on the assumption of infinite surfaces (Goodchild 1988). Potentially these studies could be connected to scaling relationships for watershed areas. This scaling relationship likely controls water color which, with lake area, is a primary factor used to compare, predict, and generalize patterns of lake ecology across northern latitudes (Seekell et al. 2018a; Seekell et al. 2018b).

Collectively, these questions indicate that there is room for fractal geometry to advance understanding of Earth’s lakes. A challenge will be developing and connecting with appropriately scaled limnological hypotheses. With a primary focus of limnology still on one lake or a few lakes in close proximity, there is still a scale miss-match to overcome for the fractal approach to reach its full potential.

References


Summary

Fractal characteristics of lakes

Fractal geometry provides a framework for understanding lake morphometry at the global scale. Here, I summarize the successes and limitations of this approach. In general, fractal geometry contributes to limnological questions that consider large populations of lakes and not individual lakes. For example, constraints on the balance of small versus large lakes are readily derived, and the total abundance and surface area of lakes can be calculated based on this relationship. Additionally, the total volume and mean depth of a population of lakes can be accurately estimated. The volume and mean depth of individual lakes cannot be estimated and this is a major limitation because most limnological questions involving these parameters are at the scale of the individual lake. I conclude by identifying some novel questions the fractal approach is suitable for addressing.